6.3 Vector potential

We found that the scalar potential function $\phi(x, y, z)$ gave us a simple way to calculate the electrostatic field of a charge distribution. If there is some charge distribution $\rho(x, y, z)$, the potential at any point (x_1, y_1, z_1) is given by the volume integral

$$\phi(x_1, y_1, z_1) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(x_2, y_2, z_2)\, dv_2}{r_{12}}. \tag{6.30}$$

The integration is extended over the whole charge distribution, and r_{12} is the magnitude of the distance from (x_2, y_2, z_2) to (x_1, y_1, z_1). The electric field $\mathbf{E}$ is obtained as the negative of the gradient of ϕ:

$$\mathbf{E} = -\text{grad}\,\phi. \tag{6.31}$$

The same trick won't work for the magnetic field, because of the essentially different character of $\mathbf{B}$. The curl of $\mathbf{B}$ is *not* necessarily zero, so $\mathbf{B}$ can't, in general, be the gradient of a scalar potential. However, we know another kind of vector derivative, the curl. It turns out that we can usefully represent $\mathbf{B}$, not as the gradient of a scalar function, but as the curl of a *vector* function, like this:

$$\boxed{\mathbf{B} = \text{curl}\,\mathbf{A}} \tag{6.32}$$

By obvious analogy, we call $\mathbf{A}$ the *vector potential.* It is *not* obvious, at this point, why this tactic is helpful. That will have to emerge as we proceed. It is encouraging that Eq. (6.28) is automatically satisfied, since $\text{div}\,\text{curl}\,\mathbf{A} = 0$, for any $\mathbf{A}$. Or, to put it another way, the fact that $\text{div}\,\mathbf{B} = 0$ presents us with the opportunity to represent $\mathbf{B}$ as the curl of another vector function.

[6] The student may wonder why we couldn't have started from some equivalent of Coulomb's law for the interaction of currents. The answer is that a piece of a current filament, unlike an electric charge, is not an independent object that can be physically isolated. You cannot perform an experiment to determine the field from *part* of a circuit; if the rest of the circuit isn't there, the current can't be steady without violating the continuity condition.

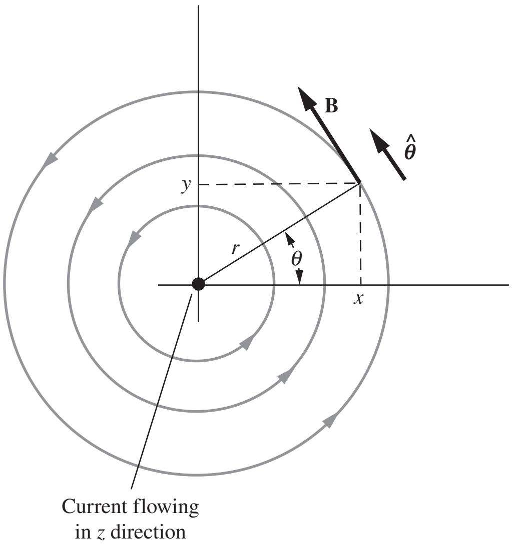


Figure 6.11.
Some field lines around a current filament. Current flows toward you (out of the plane of the paper).

Example (Vector potential for a wire) As an example of a vector potential, consider a long straight wire carrying a current I. In Fig. 6.11 we see the current coming toward us out of the page, flowing along the positive z axis. Outside the wire, what is the vector potential $\mathbf{A}$?

Solution We know what the magnetic field of the straight wire looks like. The field lines are circles, as sketched already in Fig. 6.5. A few are shown in Fig. 6.11. The magnitude of $\mathbf{B}$ is $\mu_0 I/2\pi r$. Using a unit vector $\hat{\boldsymbol{\theta}}$ in the tangential direction, we can write the vector $\mathbf{B}$ as

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\theta}}. \tag{6.33}$$

We want to find a vector field $\mathbf{A}$ whose curl equals this $\mathbf{B}$. Equation (F.2) in Appendix F gives the expression for the curl in cylindrical coordinates. In view of Eq. (6.33), we are concerned only with the $\hat{\boldsymbol{\theta}}$ component of the curl expression, which is $(\partial A_r/\partial z - \partial A_z/\partial r)\hat{\boldsymbol{\theta}}$. Due to the symmetry along the z axis, we can't have any z dependence, so we are left with only the $-(\partial A_z/\partial r)\hat{\boldsymbol{\theta}}$ term. Equating this with the $\mathbf{B}$ in Eq. (6.33) gives

$$\nabla \times \mathbf{A} = \mathbf{B} \implies -\frac{\partial A_z}{\partial r} = \frac{\mu_0 I}{2\pi r} \implies \mathbf{A} = -\hat{\mathbf{z}}\frac{\mu_0 I}{2\pi}\ln r. \tag{6.34}$$

This last step can formally be performed by separating variables and integrating. But there is no great need to do this, because we know that the integral of $1/r$ is $\ln r$. The task of Problem 6.4 is to use Cartesian coordinates to verify that the above $\mathbf{A}$ has the correct curl. See also Problem 6.5.

Of course, the $\mathbf{A}$ in Eq. (6.34) is not the only function that could serve as the vector potential for this particular $\mathbf{B}$. To this $\mathbf{A}$ could be added any vector function with zero curl. The above result holds for the space outside the wire. Inside the wire, $\mathbf{B}$ is different, so $\mathbf{A}$ must be different also. It is not hard to find the appropriate vector potential function for the interior of a solid round wire; see Exercise 6.43.

Our job now is to discover a general method of calculating $\mathbf{A}$, when the current distribution $\mathbf{J}$ is given, so that Eq. (6.32) will indeed yield the correct magnetic field. In view of Eq. (6.25), the relation between $\mathbf{J}$ and $\mathbf{A}$ is

$$\text{curl}\,(\text{curl}\,\mathbf{A}) = \mu_0 \mathbf{J}. \tag{6.35}$$

Equation (6.35), being a vector equation, is really three equations. We shall work out one of them, say the x-component equation. The x component of curl $\mathbf{B}$ is $\partial B_z/\partial y - \partial B_y/\partial z$. The z and y components of $\mathbf{B}$ are, respectively,

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}, \quad B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}. \tag{6.36}$$

Thus the x-component part of Eq. (6.35) reads

$$\frac{\partial}{\partial y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) = \mu_0 J_x. \tag{6.37}$$

We assume our functions are such that the order of partial differentiation can be interchanged. Taking advantage of that and rearranging a little, we can write Eq. (6.37) in the following way:

$$-\frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial}{\partial x}\left(\frac{\partial A_y}{\partial y}\right) + \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial z}\right) = \mu_0 J_x. \tag{6.38}$$

To make the thing more symmetrical, let's add and subtract the same term, $\partial^2 A_x/\partial x^2$, on the left:[7]

$$-\frac{\partial^2 A_x}{\partial x^2} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) = \mu_0 J_x. \tag{6.39}$$

We can now recognize the first three terms as the negative of the Laplacian of A_x. The quantity in parentheses is the divergence of $\mathbf{A}$. Now, we have a certain latitude in the construction of $\mathbf{A}$. All we care about is its curl; its divergence can be anything we like. Let us *require* that

$$\text{div}\,\mathbf{A} = 0. \tag{6.40}$$

In other words, among the various functions that might satisfy our requirement that $\text{curl}\,\mathbf{A} = \mathbf{B}$, let us consider as candidates only those that also have zero divergence. To see why we are free to do this, suppose we had an $\mathbf{A}$ such that $\text{curl}\,\mathbf{A} = \mathbf{B}$, but $\text{div}\,\mathbf{A} = f(x, y, z) \neq 0$. We claim that, for any function f, we can always find a field $\mathbf{F}$ such that $\text{curl}\,\mathbf{F} = 0$ and $\text{div}\,\mathbf{F} = -f$. If this claim is true, then we can replace $\mathbf{A}$ with the new field $\mathbf{A} + \mathbf{F}$. This field has its curl still equal to the desired value of $\mathbf{B}$, while its divergence is now equal to the desired value of zero. And the claim *is* indeed true, because if we treat $-f$ like the charge density ρ that generates an electrostatic field, we obviously can find a field $\mathbf{F}$, the analog of the electrostatic $\mathbf{E}$, such that $\text{curl}\,\mathbf{F} = 0$ and $\text{div}\,\mathbf{F} = -f$; the prescription is given in Fig. 2.29(a), without the ϵ_0.

With $\text{div}\,\mathbf{A} = 0$, the quantity in parentheses in Eq. (6.39) drops away, and we are left simply with

$$\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2} = -\mu_0 J_x, \tag{6.41}$$

where J_x is a known scalar function of x, y, z. Let us compare Eq. (6.41) with Poisson's equation, Eq. (2.73), which reads

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = -\frac{\rho}{\epsilon_0}. \tag{6.42}$$

The two equations are identical in form. We already *know* how to find a solution to Eq. (6.42). The volume integral in Eq. (6.30) is the

[7] This equation is the x component of the vector identity, $\nabla \times (\nabla \times \mathbf{A}) = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A})$. So in effect, what we've done here is prove this identity. Of course, we could have just invoked this identity and applied it to Eq. (6.35), skipping all of the intermediate steps. But it's helpful to see the proof.

prescription. Therefore a solution to Eq. (6.41) must be given by Eq. (6.30), with ρ/ϵ_0 replaced by $\mu_0 J_x$:

$$A_x(x_1, y_1, z_1) = \frac{\mu_0}{4\pi} \int \frac{J_x(x_2, y_2, z_2)\, dv_2}{r_{12}}. \tag{6.43}$$

The other components must satisfy similar formulas. They can all be combined neatly in one vector formula:

$$\boxed{\mathbf{A}(x_1, y_1, z_1) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(x_2, y_2, z_2)\, dv_2}{r_{12}}} \tag{6.44}$$

In more compact notation we have

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}\, dv}{r} \qquad \text{or} \qquad d\mathbf{A} = \frac{\mu_0}{4\pi} \frac{\mathbf{J}\, dv}{r}. \tag{6.45}$$

There is only one snag. We stipulated that $\operatorname{div} \mathbf{A} = 0$, in order to get Eq. (6.41). If the divergence of the $\mathbf{A}$ in Eq. (6.44) isn't zero, then although this $\mathbf{A}$ will satisfy Eq. (6.41), it won't satisfy Eq. (6.39). That is, it won't satisfy Eq. (6.35). Fortunately, it turns out that the $\mathbf{A}$ in Eq. (6.44) does indeed satisfy $\operatorname{div} \mathbf{A} = 0$, *provided* that the current is steady (that is, $\nabla \cdot \mathbf{J} = 0$), which is the type of situation we are concerned with. You can prove this in Problem 6.6. The proof isn't important for what we will be doing; we include it only for completeness.

Incidentally, the $\mathbf{A}$ for the example above could not have been obtained by Eq. (6.44). The integral would diverge owing to the infinite extent of the wire. This may remind you of the difficulty we encountered in Chapter 2 in setting up a scalar potential for the electric field of a charged wire. Indeed the two problems are very closely related, as we should expect from their identical geometry and the similarity of Eqs. (6.44) and (6.30). We found in Eq. (2.22) that a suitable scalar potential for the line charge problem is $-(\lambda/2\pi\epsilon_0)\ln r + C$, where C is an arbitrary constant. This assigns zero potential to some arbitrary point that is neither on the wire nor an infinite distance away. Both that scalar potential and the vector potential of Eq. (6.34) are singular at the origin and at infinity. However, see Problem 6.5 for a way to get around this issue. For an interesting discussion of the vector potential, including its interpretation as "electromagnetic momentum," see Semon and Taylor (1996).

11.3 The field of a current loop

A closed conducting loop, not necessarily circular, lies in the xy plane encircling the origin, as in Fig. 11.4(a). A steady current I flows around the loop. We are interested in the magnetic field this current creates – not near the loop, but at distant points like P_1 in the figure. We assume that r_1, the distance to P_1, is much larger than any dimension of the loop. To simplify the diagram we have located P_1 in the yz plane; it will turn out that this is no restriction. This is a good place to use the vector potential. We shall compute first the vector potential **A** at the location P_1, that is, $\mathbf{A}(0, y_1, z_1)$. From this it will be obvious what the vector potential is at any other point (x, y, z) far from the loop. Then by taking the curl of **A** we can get the magnetic field **B**.

For a current confined to a wire, Eq. (6.46) gives **A** as

$$\mathbf{A}(0, y_1, z_1) = \frac{\mu_0 I}{4\pi} \int_{\text{loop}} \frac{d\boldsymbol{l}_2}{r_{12}}. \tag{11.2}$$

When we used this equation in Section 6.4, we were concerned only with the contribution of a small segment of the circuit; now we have to integrate around the entire loop. Consider the variation in the denominator

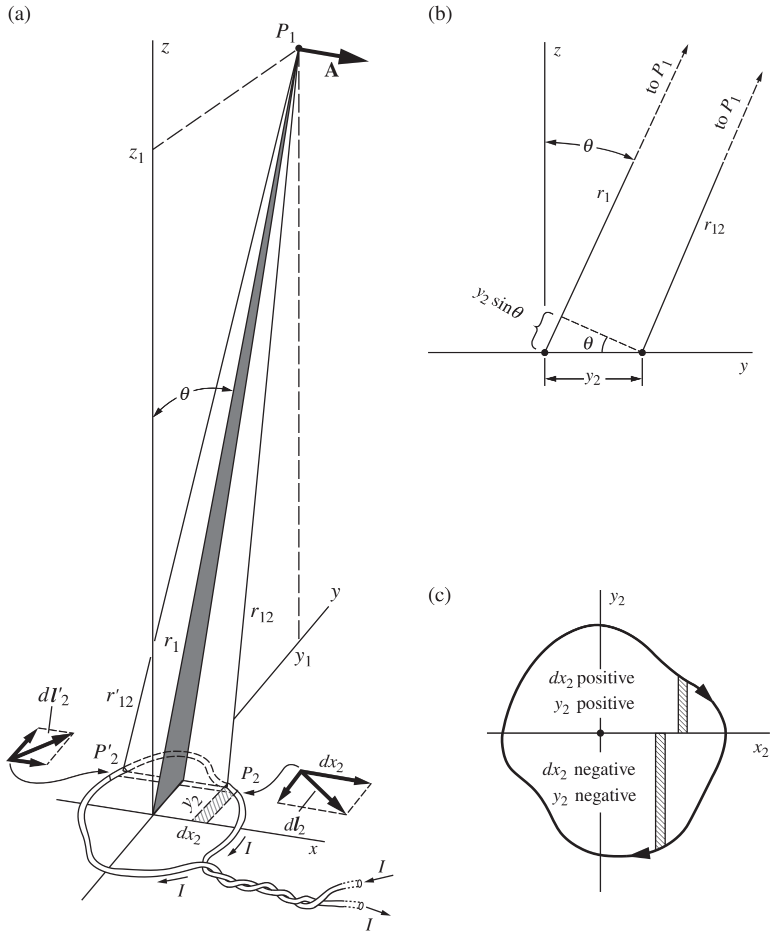


Figure 11.4.
(a) Calculation of the vector potential $\mathbf{A}$ at a point far from the current loop. (b) Side view, looking in along the x axis, showing that $r_{12} \approx r_1 - y_2 \sin\theta$ if $r_1 \gg y_2$. (c) Top view, to show that $\int_{\text{loop}} y_2\, dx_2$ is the area of the loop.

r_{12} as we go around the loop. If P_1 is far away, the first-order variation in r_{12} depends only on the coordinate y_2 of the segment $d\boldsymbol{l}_2$, and not on x_2. This is true because, from the Pythagorean theorem, the contribution to r_{12} from x_2 is of second order, whereas the side view in Fig. 11.4(b) shows the first-order contribution from y_1. Thus, neglecting quantities proportional to $(x_2/r_{12})^2$, we may treat r_{12} and r'_{12}, which lie on top of one another in the side view, as equal. And in general, to first order in the ratio (loop dimension/distance to P_1), we have

$$r_{12} \approx r_1 - y_2 \sin\theta. \tag{11.3}$$

Look now at the two elements of the path $d\boldsymbol{l}_2$ and $d\boldsymbol{l}'_2$ shown in Fig. 11.4(a). For these the dy_2 displacements are equal and opposite, and as we have already pointed out, the r_{12} distances are equal to first order. To this order then, the dy_2 contributions to the line integral will cancel, and this will be true for the whole loop. Hence **A** at P_1 will not have a y component. Obviously it will not have a z component either, for dz_2 is always zero since the current path itself has nowhere a z component.

However, **A** at P_1 *will* have an x component. The x component of the vector potential comes from the dx_2 part of the path integral:

$$\mathbf{A}(0, y_1, z_1) = \hat{\mathbf{x}}\frac{\mu_0 I}{4\pi}\int \frac{dx_2}{r_{12}}. \tag{11.4}$$

Without spoiling our first-order approximation, we can turn Eq. (11.3) into

$$\frac{1}{r_{12}} = \frac{1}{r_1\big(1 - (y_2/r_1)\sin\theta\big)} \approx \frac{1}{r_1}\left(1 + \frac{y_2 \sin\theta}{r_1}\right), \tag{11.5}$$

and using this for the integrand, we have

$$\mathbf{A}(0, y_1, z_1) = \hat{\mathbf{x}}\frac{\mu_0 I}{4\pi r_1}\int \left(1 + \frac{y_2 \sin\theta}{r_1}\right) dx_2. \tag{11.6}$$

In the integration, r_1 and θ are constants. Obviously $\int dx_2$ around the loop vanishes. Now $\int y_2\, dx_2$ around the loop is just the area of the loop, regardless of its shape; see Fig. 11.4(c). So we get finally

$$\mathbf{A}(0, y_1, z_1) = \hat{\mathbf{x}}\frac{\mu_0 I \sin\theta}{4\pi r_1^2} \times (\text{area of loop}). \tag{11.7}$$

The intuitive reason why this result is nonzero is that the parts of the loop that are closer to P_1 give larger contributions to the integral, because they have a smaller r_{12}. There is partial, but not complete, cancelation from corresponding pieces of the loop with the same x_2 value but opposite dx_2 values.

Here is a simple but crucial point: since the *shape* of the loop hasn't mattered, our restriction on P_1 to the yz plane cannot make any essential difference. Therefore we must have in Eq. (11.7) the general result we seek, if only we *state* it generally: the vector potential of a current loop of any shape, at a distance r from the loop that is much greater than the size of the loop, is a vector perpendicular to the plane containing **r** and the normal to the plane of the loop, of magnitude

$$A = \frac{\mu_0 I a \sin\theta}{4\pi r^2}, \tag{11.8}$$

where a stands for the area of the loop.

This vector potential is symmetrical around the axis of the loop, which implies that the field **B** will be symmetrical also. The explanation is that we are considering regions so far from the loop that the details of the shape of the loop have negligible influence. All loops with the same *current* × *area* product produce the same far field. We call the product Ia

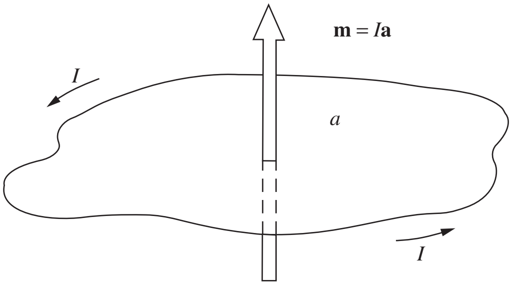


Figure 11.5.
By definition, the magnetic moment vector is related to the current by a right-hand-screw rule as shown here.

the *magnetic dipole moment* of the current loop, and denote it by **m**. Its units are amp-m^2. The magnetic dipole moment is a vector, its direction defined to be that of the normal to the loop, or that of the vector **a**, the directed area of the region surrounded by the loop:

$$\boxed{\mathbf{m} = I\mathbf{a}} \tag{11.9}$$

As for sign, let us agree that the direction of **m** and the sense of positive current flow in the loop are to be related by a right-hand-screw rule, illustrated in Fig. 11.5. (The dipole moment of the loop in Fig. 11.4(a) points downward, according to this rule.) The vector potential for the field of a magnetic dipole **m** can now be written neatly with vectors:

$$\boxed{\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}} \tag{11.10}$$

where $\hat{\mathbf{r}}$ is a unit vector in the direction *from* the loop *to* the point for which **A** is being computed. You can check that this agrees with our convention about sign. Note that the direction of **A** will always be that of the current in the *nearest* part of the loop.

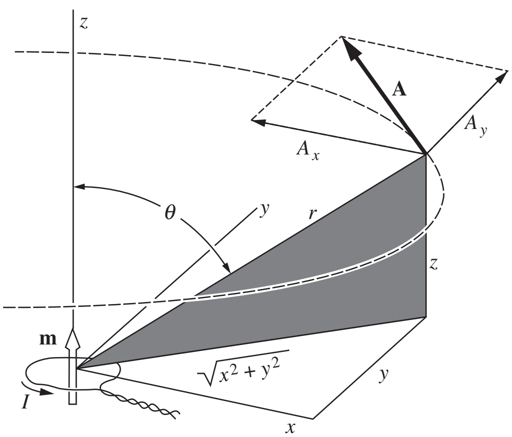


Figure 11.6.
A magnetic dipole located at the origin. At every point far from the loop, **A** is a vector parallel to the xy plane, tangent to a circle around the z axis.

Figure 11.6 shows a magnetic dipole located at the origin, with the dipole moment vector **m** pointed in the positive z direction. To express the vector potential at any point (x, y, z), we observe that $r^2 = x^2+y^2+z^2$, and $\sin\theta = \sqrt{x^2+y^2}/r$. The magnitude A of the vector potential at that point is given by

$$A = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2} = \frac{\mu_0}{4\pi}\frac{m\sqrt{x^2+y^2}}{r^3}. \tag{11.11}$$

Since **A** is tangent to a horizontal circle around the z axis, its components are

$$A_x = A\left(\frac{-y}{\sqrt{x^2+y^2}}\right) = -\frac{\mu_0}{4\pi}\frac{my}{r^3},$$

$$A_y = A\left(\frac{x}{\sqrt{x^2+y^2}}\right) = \frac{\mu_0}{4\pi}\frac{mx}{r^3},$$

$$A_z = 0. \tag{11.12}$$

Let's evaluate **B** for a point in the xz plane, by finding the components of curl **A** and then (not before!) setting $y = 0$:

$$B_x = (\nabla\times\mathbf{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -\frac{\mu_0}{4\pi}\frac{\partial}{\partial z}\frac{mx}{(x^2+y^2+z^2)^{3/2}} = \frac{\mu_0}{4\pi}\frac{3mxz}{r^5},$$

$$B_y = (\nabla\times\mathbf{A})_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = \frac{\mu_0}{4\pi}\frac{\partial}{\partial z}\frac{-my}{(x^2+y^2+z^2)^{3/2}} = \frac{\mu_0}{4\pi}\frac{3myz}{r^5},$$

$$B_z = (\nabla \times \mathbf{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$
$$= \frac{\mu_0}{4\pi} m \left[\frac{-2x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^{5/2}} + \frac{x^2 - 2y^2 + z^2}{(x^2 + y^2 + z^2)^{5/2}} \right] = \frac{\mu_0}{4\pi} \frac{m(3z^2 - r^2)}{r^5}. \tag{11.13}$$

In the xz plane, we have $y = 0$, $\sin\theta = x/r$, and $\cos\theta = z/r$. The field components at any point in that plane are thus given by

$$B_x = \frac{\mu_0}{4\pi} \frac{3m \sin\theta \cos\theta}{r^3},$$
$$B_y = 0,$$
$$B_z = \frac{\mu_0}{4\pi} \frac{m(3\cos^2\theta - 1)}{r^3}. \tag{11.14}$$

Now turn back to Section 10.3, where in Eq. (10.17) we expressed the components in the xz plane of the field $\mathbf{E}$ of an electric dipole $\mathbf{p}$, which was situated exactly like our magnetic dipole $\mathbf{m}$. The expressions are essentially identical, the only changes being $p \to m$ and $1/\epsilon_0 \to \mu_0$. We have thus found that the magnetic field of a small current loop has, at remote points, the same form as the electric field of two separated charges. We already know what that field, the electric dipole field, looks like. Figure 11.7 is an attempt to suggest the three-dimensional form of the magnetic field $\mathbf{B}$ arising from our current loop with dipole moment $\mathbf{m}$. As in the case of the electric dipole, the field is described somewhat more simply in spherical polar coordinates:

$$B_r = \frac{\mu_0 m}{2\pi r^3}\cos\theta, \quad B_\theta = \frac{\mu_0 m}{4\pi r^3}\sin\theta, \quad B_\phi = 0. \tag{11.15}$$

The magnetic field *close* to a current loop is entirely different from the electric field close to a pair of separated positive and negative charges, as the comparison in Fig. 11.8 shows. Note that between the charges the electric field points down, while inside the current ring the magnetic field points up, although the far fields are alike. This reflects the fact that our magnetic field satisfies $\nabla \cdot \mathbf{B} = 0$ everywhere, *even inside the source*. The magnetic field lines don't end. By *near* and *far* we mean, of course, relative to the size of the current loop or the separation of the charges. If we imagine the current ring shrinking in size, the current meanwhile increasing so that the dipole moment $m = Ia$ remains constant, we approach the infinitesimal magnetic dipole, the counterpart of the infinitesimal electric dipole described in Chapter 10.

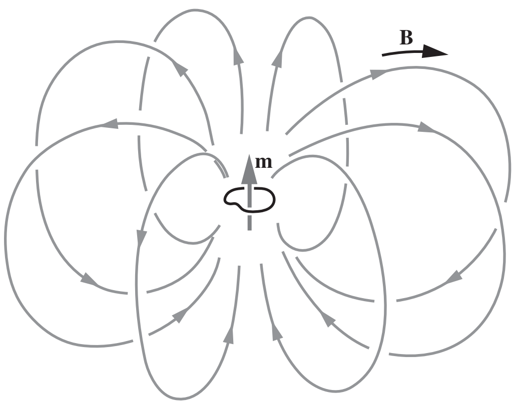


Figure 11.7.
Some magnetic field lines in the field of a magnetic dipole, that is, a small loop of current.

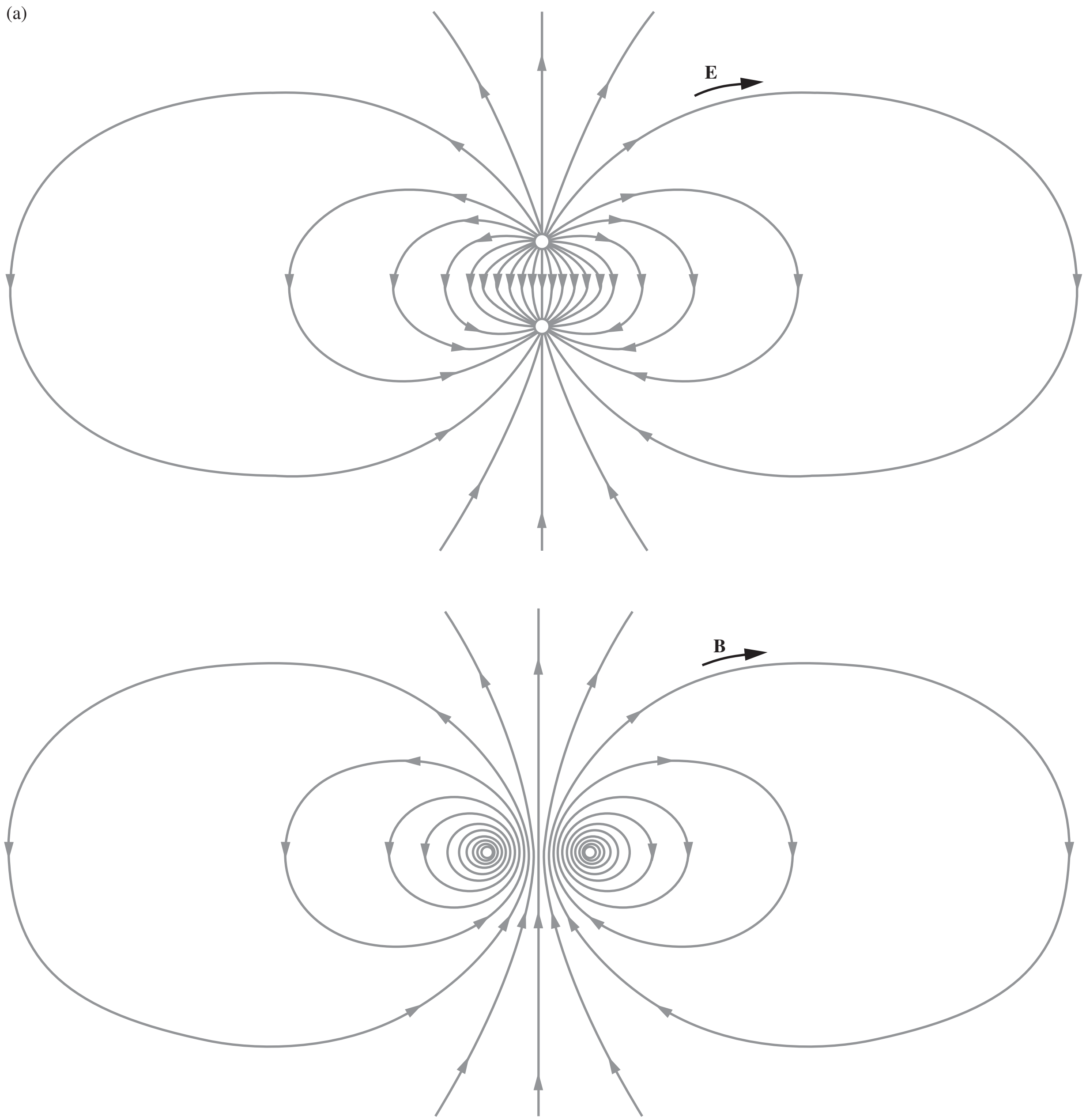


Figure 11.8.
(a) The electric field of a pair of equal and opposite charges. Far away it becomes the field of an electric dipole. (b) The magnetic field of a current ring. Far away it becomes the field of a magnetic dipole.

5.4 ■ MAGNETIC VECTOR POTENTIAL

5.4.1 ■ The Vector Potential

Just as $\nabla \times \mathbf{E} = \mathbf{0}$ permitted us to introduce a scalar potential (V) in electrostatics,

$$\mathbf{E} = -\nabla V,$$

so $\nabla \cdot \mathbf{B} = 0$ invites the introduction of a *vector* potential $\mathbf{A}$ in magnetostatics:

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}.} \tag{5.61}$$

The former is authorized by Theorem 1 (of Sect. 1.6.2), the latter by Theorem 2 (The proof of Theorem 2 is developed in Prob. 5.31). The potential formulation automatically takes care of $\nabla \cdot \mathbf{B} = 0$ (since the divergence of a curl is always zero); there remains Ampère's law:

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} = \mu_0\mathbf{J}. \tag{5.62}$$

Now, the electric potential had a built-in ambiguity: you can add to V any function whose gradient is zero (which is to say, any *constant*), without altering the *physical* quantity $\mathbf{E}$. Likewise, you can add to $\mathbf{A}$ any function whose *curl* vanishes (which is to say, the *gradient of any scalar*), with no effect on $\mathbf{B}$. We can exploit this freedom to eliminate the divergence of $\mathbf{A}$:

$$\boxed{\nabla \cdot \mathbf{A} = 0.} \tag{5.63}$$

To prove that this is always possible, suppose that our original potential, $\mathbf{A}_0$, is *not* divergenceless. If we add to it the gradient of λ ($\mathbf{A} = \mathbf{A}_0 + \nabla\lambda$), the new divergence is

$$\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A}_0 + \nabla^2\lambda.$$

We can accommodate Eq. 5.63, then, if a function λ can be found that satisfies

$$\nabla^2\lambda = -\nabla \cdot \mathbf{A}_0.$$

But this is mathematically identical to Poisson's equation (2.24),

$$\nabla^2 V = -\frac{\rho}{\epsilon_0},$$

with $\nabla \cdot \mathbf{A}_0$ in place of ρ/ϵ_0 as the "source." And we *know* how to solve Poisson's equation—that's what electrostatics is all about ("given the charge distribution, find the potential"). In particular, if ρ goes to zero at infinity, the solution is Eq. 2.29:

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho}{\mathscr{r}}\, d\tau',$$

and by the same token, if $\nabla \cdot \mathbf{A}_0$ goes to zero at infinity, then

$$\lambda = \frac{1}{4\pi}\int \frac{\nabla \cdot \mathbf{A}_0}{\mathscr{r}}\, d\tau'.$$

If $\nabla \cdot \mathbf{A}_0$ does *not* go to zero at infinity, we'll have to use other means to discover the appropriate λ, just as we get the electric potential by other means when the charge distribution extends to infinity. But the essential point remains: *It is always possible to make the vector potential divergenceless*. To put it the other way around: the definition $\mathbf{B} = \nabla \times \mathbf{A}$ specifies the *curl* of $\mathbf{A}$, but it doesn't say anything about the *divergence*—we are at liberty to pick that as we see fit, and zero is ordinarily the simplest choice.

With this condition on $\mathbf{A}$, Ampère's law (Eq. 5.62) becomes

$$\boxed{\nabla^2\mathbf{A} = -\mu_0\mathbf{J}.} \tag{5.64}$$

This *again* is nothing but Poisson's equation—or rather, it is *three* Poisson's equations, one for each Cartesian[19] component. Assuming $\mathbf{J}$ goes to zero at infinity, we can read off the solution:

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\, d\tau'.} \tag{5.65}$$

[19] In Cartesian coordinates, $\nabla^2\mathbf{A} = (\nabla^2 A_x)\hat{\mathbf{x}} + (\nabla^2 A_y)\hat{\mathbf{y}} + (\nabla^2 A_z)\hat{\mathbf{z}}$, so Eq. 5.64 reduces to $\nabla^2 A_x = -\mu_0 J_x$, $\nabla^2 A_y = -\mu_0 J_y$, and $\nabla^2 A_z = -\mu_0 J_z$. In curvilinear coordinates the unit vectors *themselves* are functions of position, and must be differentiated, so it is *not* the case, for example, that $\nabla^2 A_r = -\mu_0 J_r$. Remember that even if you plan to *evaluate* integrals such as 5.65 using curvilinear coordinates, you must first express $\mathbf{J}$ in terms of its *Cartesian* components (see Sect. 1.4.1).

For line and surface currents,

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}}{\mathscr{r}}\,dl' = \frac{\mu_0 I}{4\pi}\int \frac{1}{\mathscr{r}}\,d\mathbf{l}'; \qquad \mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}}{\mathscr{r}}\,da'. \tag{5.66}$$

(If the current does *not* go to zero at infinity, we have to find other ways to get **A**; some of these are explored in Ex. 5.12 and in the problems at the end of the section.)

It must be said that **A** is not as *useful* as V. For one thing, it's still a *vector*, and although Eqs. 5.65 and 5.66 are somewhat easier to work with than the Biot-Savart law, you still have to fuss with components. It would be nice if we could get away with a *scalar* potential

$$\mathbf{B} = -\nabla U, \tag{5.67}$$

but this is incompatible with Ampère's law, since the curl of a gradient is always zero. (A **magnetostatic scalar potential** *can* be used, if you stick scrupulously to simply-connected, current-free regions, but as a theoretical tool, it is of limited interest. See Prob. 5.29.) Moreover, since magnetic forces do no work, **A** does not admit a simple physical interpretation in terms of potential energy per unit charge. (In some contexts it can be interpreted as *momentum* per unit charge.[20]) Nevertheless, the vector potential has substantial theoretical importance, as we shall see in Chapter 10.

Example 5.11. A spherical shell of radius R, carrying a uniform surface charge σ, is set spinning at angular velocity $\boldsymbol{\omega}$. Find the vector potential it produces at point $\mathbf{r}$ (Fig. 5.45).

Solution

It might seem natural to set the polar axis along $\boldsymbol{\omega}$, but in fact the integration is easier if we let $\mathbf{r}$ lie on the z axis, so that $\boldsymbol{\omega}$ is tilted at an angle ψ. We may as well orient the x axis so that $\boldsymbol{\omega}$ lies in the xz plane, as shown in Fig. 5.46. According to Eq. 5.66,

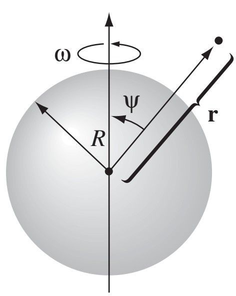


FIGURE 5.45

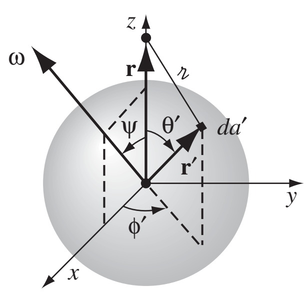


FIGURE 5.46

[20]M. D. Semon and J. R. Taylor, *Am. J. Phys.* **64**, 1361 (1996).

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}')}{\mathscr{r}}\,da',$$

where $\mathbf{K} = \sigma\mathbf{v}$, $\mathscr{r} = \sqrt{R^2 + r^2 - 2Rr\cos\theta'}$, and $da' = R^2\sin\theta'\,d\theta'\,d\phi'$. Now the velocity of a point $\mathbf{r}'$ in a rotating rigid body is given by $\boldsymbol{\omega}\times\mathbf{r}'$; in this case,

$$\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}' = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \omega\sin\psi & 0 & \omega\cos\psi \\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\theta' \end{vmatrix}$$

$$= R\omega\left[-\left(\cos\psi\,\sin\theta'\sin\phi'\right)\hat{\mathbf{x}} + (\cos\psi\,\sin\theta'\cos\phi' - \sin\psi\,\cos\theta')\,\hat{\mathbf{y}}\right.$$

$$\left. + \,(\sin\psi\,\sin\theta'\sin\phi')\,\hat{\mathbf{z}}\right].$$

Notice that each of these terms, save one, involves either $\sin\phi'$ or $\cos\phi'$. Since

$$\int_0^{2\pi}\sin\phi'\,d\phi' = \int_0^{2\pi}\cos\phi'\,d\phi' = 0,$$

such terms contribute nothing. There remains

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 R^3\sigma\omega\sin\psi}{2}\left(\int_0^{\pi}\frac{\cos\theta'\sin\theta'}{\sqrt{R^2+r^2-2Rr\cos\theta'}}\,d\theta'\right)\hat{\mathbf{y}}.$$

Letting $u \equiv \cos\theta'$, the integral becomes

$$\int_{-1}^{+1}\frac{u}{\sqrt{R^2+r^2-2Rru}}\,du = -\frac{(R^2+r^2+Rru)}{3R^2r^2}\sqrt{R^2+r^2-2Rru}\,\Bigg|_{-1}^{+1}$$

$$= -\frac{1}{3R^2r^2}\left[(R^2+r^2+Rr)|R-r|\right.$$

$$\left. -(R^2+r^2-Rr)(R+r)\right].$$

If the point $\mathbf{r}$ lies *inside* the sphere, then $R > r$, and this expression reduces to $(2r/3R^2)$; if $\mathbf{r}$ lies *outside* the sphere, so that $R < r$, it reduces to $(2R/3r^2)$. Noting that $(\boldsymbol{\omega}\times\mathbf{r}) = -\omega r\sin\psi\,\hat{\mathbf{y}}$, we have, finally,

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0 R\sigma}{3}(\boldsymbol{\omega}\times\mathbf{r}), & \text{for points } \textit{inside} \text{ the sphere,} \\ \dfrac{\mu_0 R^4\sigma}{3r^3}(\boldsymbol{\omega}\times\mathbf{r}), & \text{for points } \textit{outside} \text{ the sphere.} \end{cases} \quad (5.68)$$

Having evaluated the integral, I revert to the "natural" coordinates of Fig. 5.45, in which $\boldsymbol{\omega}$ coincides with the z axis and the point $\mathbf{r}$ is at (r, θ, ϕ):

$$\mathbf{A}(r,\theta,\phi) = \begin{cases} \dfrac{\mu_0 R\omega\sigma}{3} r\sin\theta\,\hat{\boldsymbol{\phi}}, & (r \le R), \\ \dfrac{\mu_0 R^4\omega\sigma}{3}\dfrac{\sin\theta}{r^2}\,\hat{\boldsymbol{\phi}}, & (r \ge R). \end{cases} \quad (5.69)$$

Curiously, the field inside this spherical shell is *uniform*:

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{2\mu_0 R\omega\sigma}{3}(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) = \frac{2}{3}\mu_0\sigma R\omega\,\hat{\mathbf{z}} = \frac{2}{3}\mu_0\sigma R\boldsymbol{\omega}. \quad (5.70)$$

Example 5.12. Find the vector potential of an infinite solenoid with n turns per unit length, radius R, and current I.

Solution

This time we cannot use Eq. 5.66, since the current itself extends to infinity. But here's a cute method that does the job. Notice that

$$\oint \mathbf{A}\cdot d\mathbf{l} = \int(\nabla \times \mathbf{A})\cdot d\mathbf{a} = \int \mathbf{B}\cdot d\mathbf{a} = \Phi, \quad (5.71)$$

where Φ is the flux of $\mathbf{B}$ through the loop in question. This is reminiscent of Ampère's law in integral form (Eq. 5.57),

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{\text{enc}}.$$

In fact, it's the same equation, with $\mathbf{B} \to \mathbf{A}$ and $\mu_0 I_{\text{enc}} \to \Phi$. If symmetry permits, we can determine $\mathbf{A}$ from Φ in the same way we got $\mathbf{B}$ from I_{enc}, in Sect. 5.3.3. The present problem (with a uniform longitudinal magnetic field $\mu_0 nI$ inside the solenoid and no field outside) is analogous to the Ampère's law problem of a fat wire carrying a uniformly distributed current. The vector potential is "circumferential" (it mimics the magnetic field in the analog); using a circular "Amperian loop" at radius s *inside* the solenoid, we have

$$\oint \mathbf{A}\cdot d\mathbf{l} = A(2\pi s) = \int \mathbf{B}\cdot d\mathbf{a} = \mu_0 nI(\pi s^2),$$

so

$$\mathbf{A} = \frac{\mu_0 nI}{2} s\,\hat{\boldsymbol{\phi}}, \quad \text{for } s \le R. \quad (5.72)$$

For an Amperian loop *outside* the solenoid, the flux is

$$\int \mathbf{B}\cdot d\mathbf{a} = \mu_0 nI(\pi R^2),$$

since the field only extends out to R. Thus

$$\mathbf{A} = \frac{\mu_0 nI}{2}\frac{R^2}{s}\,\hat{\boldsymbol{\phi}}, \quad \text{for } s \ge R. \quad (5.73)$$

If you have any doubts about this answer, *check* it: Does $\nabla \times \mathbf{A} = \mathbf{B}$? Does $\nabla \cdot \mathbf{A} = 0$? If so, we're done.

Typically, the direction of **A** mimics the direction of the current. For instance, both were azimuthal in Exs. 5.11 and 5.12. Indeed, if all the current flows in *one* direction, then Eq. 5.65 suggests that **A** *must* point that way too. Thus the potential of a finite segment of straight wire (Prob. 5.23) is in the direction of the current. Of course, if the current extends to infinity you can't use Eq. 5.65 in the first place (see Probs. 5.26 and 5.27). Moreover, you can always add an arbitrary constant vector to **A**—this is analogous to changing the reference point for V, and it won't affect the divergence or curl of **A**, which is all that matters (in Eq. 5.65 we have chosen the constant so that **A** goes to zero at infinity). In principle you could even use a vector potential that is not divergenceless, in which case all bets are off. Despite these caveats, the essential point remains: *Ordinarily* the direction of **A** will match the direction of the current.

Problem 5.23 Find the magnetic vector potential of a finite segment of straight wire carrying a current I. [Put the wire on the z axis, from z_1 to z_2, and use Eq. 5.66.] Check that your answer is consistent with Eq. 5.37.

Problem 5.24 What current density would produce the vector potential, $\mathbf{A} = k\,\hat{\boldsymbol{\phi}}$ (where k is a constant), in cylindrical coordinates?

Problem 5.25 If **B** is *uniform*, show that $\mathbf{A}(\mathbf{r}) = -\frac{1}{2}(\mathbf{r} \times \mathbf{B})$ works. That is, check that $\nabla \cdot \mathbf{A} = 0$ and $\nabla \times \mathbf{A} = \mathbf{B}$. Is this result unique, or are there other functions with the same divergence and curl?

Problem 5.26

(a) By whatever means you can think of (short of looking it up), find the vector potential a distance s from an infinite straight wire carrying a current I. Check that $\nabla \cdot \mathbf{A} = 0$ and $\nabla \times \mathbf{A} = \mathbf{B}$.

(b) Find the magnetic potential *inside* the wire, if it has radius R and the current is uniformly distributed.

Problem 5.27 Find the vector potential above and below the plane surface current in Ex. 5.8.

Problem 5.28

(a) Check that Eq. 5.65 is consistent with Eq. 5.63, by applying the *divergence*.

(b) Check that Eq. 5.65 is consistent with Eq. 5.47, by applying the *curl*.

(c) Check that Eq. 5.65 is consistent with Eq. 5.64, by applying the *Laplacian*.

Problem 5.29 Suppose you want to define a magnetic scalar potential U (Eq. 5.67) in the vicinity of a current-carrying wire. First of all, you must stay away from the wire itself (there $\nabla \times \mathbf{B} \neq \mathbf{0}$); but that's not enough. Show, by applying Ampère's law to a path that starts at **a** and circles the wire, returning to **b** (Fig. 5.47), that the scalar potential cannot be single-valued (that is, $U(\mathbf{a}) \neq U(\mathbf{b})$, even if they represent the same physical point). As an example, find the scalar potential for an infinite

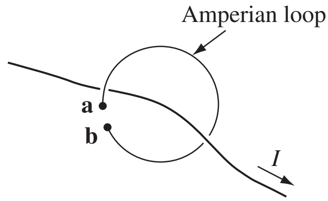


FIGURE 5.47

straight wire. (To avoid a multivalued potential, you must restrict yourself to simply-connected regions that remain on one side or the other of every wire, never allowing you to go all the way around.)

Problem 5.30 Use the results of Ex. 5.11 to find the magnetic field inside a solid sphere, of uniform charge density ρ and radius R, that is rotating at a constant angular velocity $\boldsymbol{\omega}$.

Problem 5.31

(a) Complete the proof of Theorem 2, Sect. 1.6.2. That is, show that any divergenceless vector field $\mathbf{F}$ can be written as the curl of a vector potential $\mathbf{A}$. What you have to do is find A_x, A_y, and A_z such that (i) $\partial A_z/\partial y - \partial A_y/\partial z = F_x$; (ii) $\partial A_x/\partial z - \partial A_z/\partial x = F_y$; and (iii) $\partial A_y/\partial x - \partial A_x/\partial y = F_z$. Here's one way to do it: Pick $A_x = 0$, and solve (ii) and (iii) for A_y and A_z. Note that the "constants of integration" are themselves functions of y and z—they're constant only with respect to x. Now plug these expressions into (i), and use the fact that $\nabla \cdot \mathbf{F} = 0$ to obtain

$$A_y = \int_0^x F_z(x', y, z)\, dx'; \quad A_z = \int_0^y F_x(0, y', z)\, dy' - \int_0^x F_y(x', y, z)\, dx'.$$

(b) By direct differentiation, check that the $\mathbf{A}$ you obtained in part (a) satisfies $\nabla \times \mathbf{A} = \mathbf{F}$. Is $\mathbf{A}$ divergenceless? [This was a very asymmetrical construction, and it would be surprising if it *were*—although we know that there *exists* a vector whose curl is $\mathbf{F}$ *and* whose divergence is zero.]

(c) As an example, let $\mathbf{F} = y\,\hat{\mathbf{x}} + z\,\hat{\mathbf{y}} + x\,\hat{\mathbf{z}}$. Calculate $\mathbf{A}$, and confirm that $\nabla \times \mathbf{A} = \mathbf{F}$. (For further discussion, see Prob. 5.53.)

5.4.2 ■ Boundary Conditions

In Chapter 2, I drew a triangular diagram to summarize the relations among the three fundamental quantities of electrostatics: the charge density ρ, the electric field $\mathbf{E}$, and the potential V. A similar figure can be constructed for magnetostatics (Fig. 5.48), relating the current density $\mathbf{J}$, the field $\mathbf{B}$, and the potential $\mathbf{A}$. There is one "missing link" in the diagram: the equation for $\mathbf{A}$ in terms of $\mathbf{B}$. It's unlikely you would ever need such a formula, but in case you are interested, see Probs. 5.52 and 5.53.

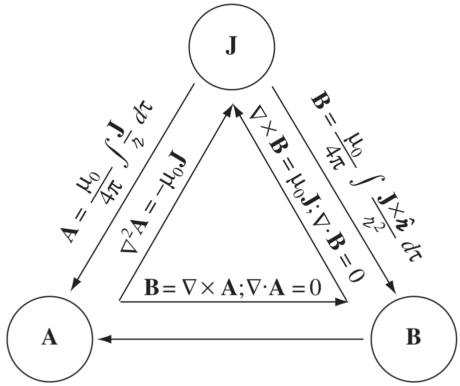


FIGURE 5.48

Just as the electric field suffers a discontinuity at a surface *charge*, so the magnetic field is discontinuous at a surface *current*. Only this time it is the *tangential* component that changes. For if we apply Eq. 5.50, in integral form,

$$\oint \mathbf{B}\cdot d\mathbf{a} = 0,$$

to a wafer-thin pillbox straddling the surface (Fig. 5.49), we get

$$B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}}. \tag{5.74}$$

As for the tangential components, an Amperian loop running perpendicular to the current (Fig. 5.50) yields

$$\oint \mathbf{B}\cdot d\mathbf{l} = \left(B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}}\right) l = \mu_0 I_{\text{enc}} = \mu_0 K l,$$

or

$$B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}} = \mu_0 K. \tag{5.75}$$

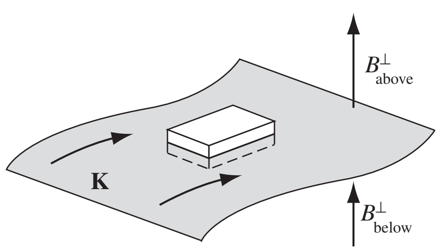


FIGURE 5.49

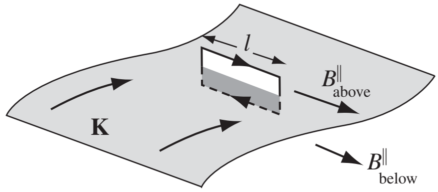


FIGURE 5.50

Thus the component of **B** that is parallel to the surface but perpendicular to the current is discontinuous in the amount $\mu_0 K$. A similar Amperian loop running *parallel* to the current reveals that the *parallel* component is *continuous*. These results can be summarized in a single formula:

$$\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0(\mathbf{K} \times \hat{\mathbf{n}}), \tag{5.76}$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the surface, pointing "upward."

Like the scalar potential in electrostatics, the vector potential is continuous across any boundary:

$$\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}}, \tag{5.77}$$

for $\nabla \cdot \mathbf{A} = 0$ guarantees[21] that the *normal* component is continuous; and $\nabla \times \mathbf{A} = \mathbf{B}$, in the form

$$\oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{a} = \Phi,$$

means that the tangential components are continuous (the flux through an Amperian loop of vanishing thickness is zero). But the *derivative* of **A** inherits the discontinuity of **B**:

$$\frac{\partial \mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial \mathbf{A}_{\text{below}}}{\partial n} = -\mu_0 \mathbf{K}. \tag{5.78}$$

Problem 5.32

(a) Check Eq. 5.76 for the configuration in Ex. 5.9.

(b) Check Eqs. 5.77 and 5.78 for the configuration in Ex. 5.11.

Problem 5.33 Prove Eq. 5.78, using Eqs. 5.63, 5.76, and 5.77. [*Suggestion:* I'd set up Cartesian coordinates at the surface, with z perpendicular to the surface and x parallel to the current.]

---

[21] Note that Eqs. 5.77 and 5.78 presuppose that **A** is divergenceless.

5.4.3 ■ Multipole Expansion of the Vector Potential

If you want an approximate formula for the vector potential of a localized current distribution, valid at distant points, a multipole expansion is in order. Remember: the idea of a multipole expansion is to write the potential in the form of a power series in $1/r$, where r is the distance to the point in question (Fig. 5.51); if r is sufficiently large, the series will be dominated by the lowest nonvanishing contribution, and the higher terms can be ignored. As we found in Sect. 3.4.1 (Eq. 3.94),

$$\frac{1}{\mathscr{r}} = \frac{1}{\sqrt{r^2 + (r')^2 - 2rr'\cos\alpha}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha), \tag{5.79}$$

where α is the angle between $\mathbf{r}$ and $\mathbf{r}'$. Accordingly, the vector potential of a current loop can be written

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\oint \frac{1}{\mathscr{r}}\, d\mathbf{l}' = \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint (r')^n P_n(\cos\alpha)\, d\mathbf{l}', \tag{5.80}$$

or, more explicitly:

$$\begin{aligned}\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\bigg[&\frac{1}{r}\oint d\mathbf{l}' + \frac{1}{r^2}\oint r'\cos\alpha\, d\mathbf{l}' \\ &+ \frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right) d\mathbf{l}' + \cdots\bigg].\end{aligned} \tag{5.81}$$

As in the multipole expansion of V, we call the first term (which goes like $1/r$) the **monopole** term, the second (which goes like $1/r^2$) **dipole**, the third **quadrupole**, and so on.

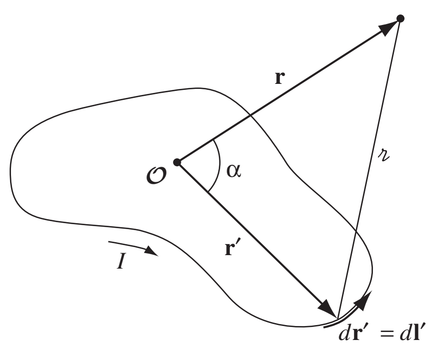


FIGURE 5.51

Now, the *magnetic monopole term is always zero*, for the integral is just the total vector displacement around a closed loop:

$$\oint d\mathbf{l}' = \mathbf{0}. \tag{5.82}$$

This reflects the fact that there are no magnetic monopoles in nature (an assumption contained in Maxwell's equation $\nabla \cdot \mathbf{B} = 0$, on which the entire theory of vector potential is predicated).

In the absence of any monopole contribution, the dominant term is the dipole (except in the rare case where it, too, vanishes):

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi r^2} \oint r' \cos\alpha \, d\mathbf{l}' = \frac{\mu_0 I}{4\pi r^2} \oint (\hat{\mathbf{r}} \cdot \mathbf{r}') \, d\mathbf{l}'. \tag{5.83}$$

This integral can be rewritten in a more illuminating way if we invoke Eq. 1.108, with $\mathbf{c} = \hat{\mathbf{r}}$:

$$\oint (\hat{\mathbf{r}} \cdot \mathbf{r}') \, d\mathbf{l}' = -\hat{\mathbf{r}} \times \int d\mathbf{a}'. \tag{5.84}$$

Then

$$\boxed{\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2},} \tag{5.85}$$

where $\mathbf{m}$ is the **magnetic dipole moment**:

$$\boxed{\mathbf{m} \equiv I \int d\mathbf{a} = I\mathbf{a}.} \tag{5.86}$$

Here $\mathbf{a}$ is the "vector area" of the loop (Prob. 1.62); if the loop is *flat*, $\mathbf{a}$ is the ordinary area enclosed, with the direction assigned by the usual right-hand rule (fingers in the direction of the current).

Example 5.13. Find the magnetic dipole moment of the "bookend-shaped" loop shown in Fig. 5.52. All sides have length w, and it carries a current I.

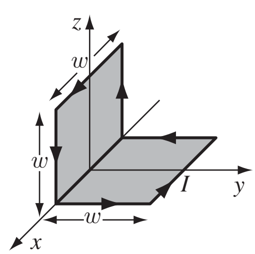


FIGURE 5.52

Solution

This wire could be considered the superposition of two plane square loops (Fig. 5.53). The "extra" sides (AB) cancel when the two are put together, since the currents flow in opposite directions. The net magnetic dipole moment is

$$\mathbf{m} = Iw^2\,\hat{\mathbf{y}} + Iw^2\,\hat{\mathbf{z}};$$

its magnitude is $\sqrt{2}Iw^2$, and it points along the 45° line $z = y$.

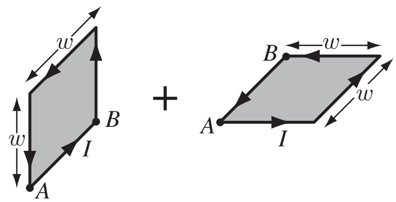


FIGURE 5.53

It is clear from Eq. 5.86 that the magnetic dipole moment is independent of the choice of origin. You may remember that the *electric* dipole moment is independent of the origin only when the total charge vanishes (Sect. 3.4.3). Since the *magnetic* monopole moment is *always* zero, it is not really surprising that the magnetic dipole moment is always independent of origin.

Although the dipole term *dominates* the multipole expansion (unless $\mathbf{m} = 0$) and thus offers a good approximation to the true potential, it is not ordinarily the *exact* potential; there will be quadrupole, octopole, and higher contributions. You might ask, is it possible to devise a current distribution whose potential is "pure" dipole—for which Eq. 5.85 is *exact*? Well, yes and no: like the electrical analog, it can be done, but the model is a bit contrived. To begin with, you must take an *infinitesimally small* loop at the origin, but then, in order to keep the dipole moment finite, you have to crank the current up to infinity, with the product $m = Ia$ held fixed. In practice, the dipole potential is a suitable approximation whenever the distance r greatly exceeds the size of the loop.

The magnetic *field* of a (perfect) dipole is easiest to calculate if we put $\mathbf{m}$ at the origin and let it point in the z-direction (Fig. 5.54). According to Eq. 5.85, the potential at point (r, θ, ϕ) is

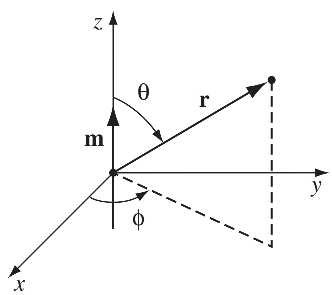
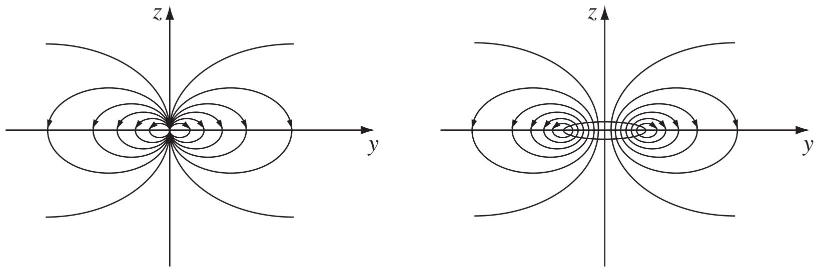


FIGURE 5.54



(a) Field of a "pure" dipole (b) Field of a "physical" dipole

FIGURE 5.55

$$\mathbf{A}_{\rm dip}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\,\hat{\boldsymbol{\phi}}, \tag{5.87}$$

and hence

$$\mathbf{B}_{\rm dip}(\mathbf{r}) = \nabla\times\mathbf{A} = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}). \tag{5.88}$$

Surprisingly, this is *identical* in structure to the field of an *electric* dipole (Eq. 3.103)! (Up close, however, the field of a *physical* magnetic dipole—a small current loop—looks quite different from the field of a physical electric dipole—plus and minus charges a short distance apart. Compare Fig. 5.55 with Fig. 3.37.)

• **Problem 5.34** Show that the magnetic field of a dipole can be written in coordinate-free form:

$$\boxed{\mathbf{B}_{\rm dip}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\right].} \tag{5.89}$$

Problem 5.35 A circular loop of wire, with radius R, lies in the xy plane (centered at the origin) and carries a current I running counterclockwise as viewed from the positive z axis.

(a) What is its magnetic dipole moment?

(b) What is the (approximate) magnetic field at points far from the origin?

(c) Show that, for points on the z axis, your answer is consistent with the *exact* field (Ex. 5.6), when $z \gg R$.

Problem 5.36 Find the exact magnetic field a distance z above the center of a square loop of side w, carrying a current I. Verify that it reduces to the field of a dipole, with the appropriate dipole moment, when $z \gg w$.

Problem 5.37

(a) A phonograph record of radius R, carrying a uniform surface charge σ, is rotating at constant angular velocity ω. Find its magnetic dipole moment.

(b) Find the magnetic dipole moment of the spinning spherical shell in Ex. 5.11. Show that for points $r > R$ the potential is that of a perfect dipole.

Problem 5.38 I worked out the multipole expansion for the vector potential of a *line* current because that's the most common type, and in some respects the easiest to handle. For a *volume* current $\mathbf{J}$:

(a) Write down the multipole expansion, analogous to Eq. 5.80.

(b) Write down the monopole potential, and prove that it vanishes.

(c) Using Eqs. 1.107 and 5.86, show that the dipole moment can be written

$$\mathbf{m} = \frac{1}{2}\int (\mathbf{r} \times \mathbf{J})\, d\tau. \tag{5.90}$$

More Problems on Chapter 5

Problem 5.39 Analyze the motion of a particle (charge q, mass m) in the magnetic field of a long straight wire carrying a steady current I.

(a) Is its kinetic energy conserved?

(b) Find the force on the particle, in cylindrical coordinates, with I along the z axis.

(c) Obtain the equations of motion.

(d) Suppose $\dot{z}$ is constant. Describe the motion.

Problem 5.40 It may have occurred to you that since parallel currents attract, the current within a single wire should contract into a tiny concentrated stream along the axis. Yet in practice the current typically distributes itself quite uniformly over the wire. How do you account for this? If the positive charges (density ρ_+) are "nailed down," and the negative charges (density ρ_-) move at speed v (and none of these depends on the distance from the axis), show that $\rho_- = -\rho_+\gamma^2$, where $\gamma \equiv 1/\sqrt{1-(v/c)^2}$ and $c^2 = 1/\mu_0\epsilon_0$. If the wire as a whole is neutral, where is the compensating charge located?[22] [Notice that for typical velocities (see Prob. 5.20), the two charge densities are essentially unchanged by the current (since $\gamma \approx 1$). In **plasmas**, however, where the positive charges are *also* free to move, this so-called **pinch effect** can be very significant.]

Problem 5.41 A current I flows to the right through a rectangular bar of conducting material, in the presence of a uniform magnetic field $\mathbf{B}$ pointing out of the page (Fig. 5.56).

(a) If the moving charges are *positive*, in which direction are they deflected by the magnetic field? This deflection results in an accumulation of charge on the

[22] For further discussion, see D. C. Gabuzda, *Am. J. Phys.* **61**, 360 (1993).

upper and lower surfaces of the bar, which in turn produces an electric force to counteract the magnetic one. Equilibrium occurs when the two exactly cancel. (This phenomenon is known as the **Hall effect**.)

(b) Find the resulting potential difference (the **Hall voltage**) between the top and bottom of the bar, in terms of B, v (the speed of the charges), and the relevant dimensions of the bar.[23]

(c) How would your analysis change if the moving charges were *negative*? [The Hall effect is the classic way of determining the sign of the mobile charge carriers in a material.]

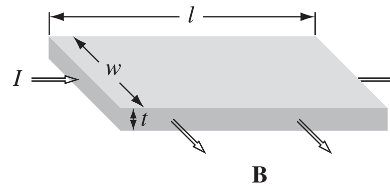


FIGURE 5.56

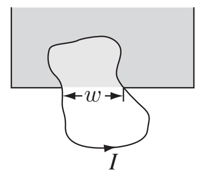


FIGURE 5.57

Problem 5.42 A plane wire loop of irregular shape is situated so that part of it is in a uniform magnetic field **B** (in Fig. 5.57 the field occupies the shaded region, and points perpendicular to the plane of the loop). The loop carries a current I. Show that the net magnetic force on the loop is $F = IBw$, where w is the chord subtended. Generalize this result to the case where the magnetic field region itself has an irregular shape. What is the direction of the force?

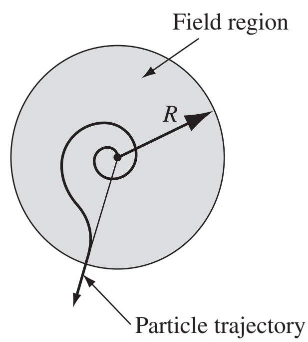


FIGURE 5.58

Problem 5.43 A circularly symmetrical magnetic field (**B** depends only on the distance from the axis), pointing perpendicular to the page, occupies the shaded region in Fig. 5.58. If the total flux ($\int \mathbf{B} \cdot d\mathbf{a}$) is zero, show that a charged particle that starts out at the center will emerge from the field region on a *radial* path (provided

[23]The potential *within* the bar makes an interesting boundary-value problem. See M. J. Moelter, J. Evans, G. Elliot, and M. Jackson, *Am. J. Phys.* **66**, 668 (1998).

it escapes at all). On the reverse trajectory, a particle fired at the center from outside will hit its target (if it has sufficient energy), though it may follow a weird route getting there. [*Hint:* Calculate the total angular momentum acquired by the particle, using the Lorentz force law.]

Problem 5.44 Calculate the magnetic force of attraction between the northern and southern hemispheres of a spinning charged spherical shell (Ex. 5.11). [*Answer:* $(\pi/4)\mu_0\sigma^2\omega^2 R^4$.]

! **Problem 5.45** Consider the motion of a particle with mass m and electric charge q_e in the field of a (hypothetical) stationary magnetic *monopole* q_m at the origin:

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{q_m}{r^2}\,\hat{\mathbf{r}}.$$

(a) Find the acceleration of q_e, expressing your answer in terms of q, q_m, m, $\mathbf{r}$ (the position of the particle), and $\mathbf{v}$ (its velocity).

(b) Show that the speed $v = |\mathbf{v}|$ is a constant of the motion.

(c) Show that the vector quantity

$$\mathbf{Q} \equiv m(\mathbf{r} \times \mathbf{v}) - \frac{\mu_0 q_e q_m}{4\pi}\,\hat{\mathbf{r}}$$

is a constant of the motion. [*Hint:* differentiate it with respect to time, and prove—using the equation of motion from (a)—that the derivative is zero.]

(d) Choosing spherical coordinates (r, θ, ϕ), with the polar (z) axis along $\mathbf{Q}$,

(i) calculate $\mathbf{Q}\cdot\hat{\boldsymbol{\phi}}$, and show that θ is a constant of the motion (so q_e moves on the surface of a cone—something Poincaré first discovered in 1896)[24];

(ii) calculate $\mathbf{Q}\cdot\hat{\mathbf{r}}$, and show that the magnitude of $\mathbf{Q}$ is

$$Q = \frac{\mu_0}{4\pi}\left|\frac{q_e q_m}{\cos\theta}\right|;$$

(iii) calculate $\mathbf{Q}\cdot\hat{\boldsymbol{\theta}}$, show that

$$\frac{d\phi}{dt} = \frac{k}{r^2},$$

and determine the constant k.

(e) By expressing v^2 in spherical coordinates, obtain the equation for the trajectory, in the form

$$\frac{dr}{d\phi} = f(r)$$

(that is: determine the function $f(r)$).

(f) Solve this equation for $r(\phi)$.

[24] In point of fact, the charge follows a *geodesic* on the cone. The original paper is H. Poincaré, *Comptes rendus de l'Academie des Sciences* **123**, 530 (1896); for a more modern treatment, see B. Rossi and S. Olbert, *Introduction to the Physics of Space* (New York: McGraw-Hill, 1970).

! **Problem 5.46** Use the Biot-Savart law (most conveniently in the form of Eq. 5.42 appropriate to surface currents) to find the field inside and outside an infinitely long solenoid of radius R, with n turns per unit length, carrying a steady current I.

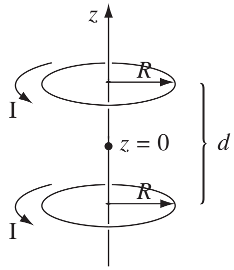


FIGURE 5.59

Problem 5.47 The magnetic field on the axis of a circular current loop (Eq. 5.41) is far from uniform (it falls off sharply with increasing z). You can produce a more nearly uniform field by using *two* such loops a distance d apart (Fig. 5.59).

(a) Find the field (B) as a function of z, and show that $\partial B/\partial z$ is zero at the point midway between them ($z = 0$).

(b) If you pick d just right, the *second* derivative of B will *also* vanish at the midpoint. This arrangement is known as a **Helmholtz coil**; it's a convenient way of producing relatively uniform fields in the laboratory. Determine d such that $\partial^2 B/\partial z^2 = 0$ at the midpoint, and find the resulting magnetic field at the center. [*Answer:* $8\mu_0 I/5\sqrt{5}R$]

Problem 5.48 Use Eq. 5.41 to obtain the magnetic field on the axis of the rotating disk in Prob. 5.37(a). Show that the dipole field (Eq. 5.88), with the dipole moment you found in Prob. 5.37, is a good approximation if $z \gg R$.

Problem 5.49 Suppose you wanted to find the field of a circular loop (Ex. 5.6) at a point $\mathbf{r}$ that is *not* directly above the center (Fig. 5.60). You might as well choose your axes so that $\mathbf{r}$ lies in the yz plane at $(0, y, z)$. The source point is $(R\cos\phi', R\sin\phi', 0)$, and ϕ' runs from 0 to 2π. Set up the integrals[25] from which you could calculate B_x, B_y, and B_z, and evaluate B_x explicitly.

Problem 5.50 Magnetostatics treats the "source current" (the one that sets up the field) and the "recipient current" (the one that experiences the force) so asymmetrically that it is by no means obvious that the magnetic force between two current loops is consistent with Newton's third law. Show, starting with the Biot-Savart law (Eq. 5.34) and the Lorentz force law (Eq. 5.16), that the force on loop 2 due to loop 1 (Fig. 5.61) can be written as

$$\mathbf{F}_2 = -\frac{\mu_0}{4\pi} I_1 I_2 \oint \oint \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\mathbf{l}_1 \cdot d\mathbf{l}_2. \tag{5.91}$$

[25]These are **elliptic integrals**—see R. H. Good, *Eur. J. Phys.* **22**, 119 (2001).

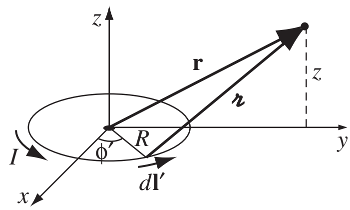


FIGURE 5.60

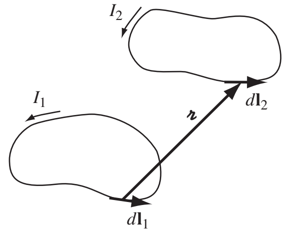


FIGURE 5.61

In this form, it is clear that $\mathbf{F}_2 = -\mathbf{F}_1$, since $\hat{\boldsymbol{\mathscr{r}}}$ changes direction when the roles of 1 and 2 are interchanged. (If you seem to be getting an "extra" term, it will help to note that $d\mathbf{l}_2 \cdot \hat{\boldsymbol{\mathscr{r}}} = d\mathscr{r}$.)

Problem 5.51 Consider a *plane* loop of wire that carries a steady current I; we want to calculate the magnetic field at a point in the plane. We might as well take that point to be the origin (it could be inside or outside the loop). The shape of the wire is given, in polar coordinates, by a specified function $r(\theta)$ (Fig. 5.62).

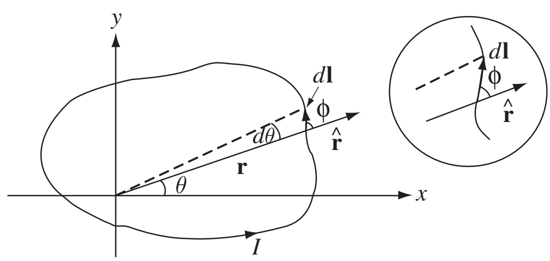


FIGURE 5.62

(a) Show that the magnitude of the field is[26]

$$B = \frac{\mu_0 I}{4\pi} \oint \frac{d\theta}{r}. \tag{5.92}$$

[*Hint:* Start with the Biot-Savart law; note that $\boldsymbol{\mathscr{r}} = -\mathbf{r}$, and $d\mathbf{l} \times \hat{\mathbf{r}}$ points perpendicular to the plane; show that $|d\mathbf{l} \times \hat{\mathbf{r}}| = dl\, \sin\phi = r\, d\theta$.]

(b) Test this formula by calculating the field at the center of a circular loop.

(c) The "lituus spiral" is defined by

$$r(\theta) = \frac{a}{\sqrt{\theta}}, \quad (0 < \theta \leq 2\pi)$$

(for some constant a). Sketch this figure, and complete the loop with a straight segment along the x axis. What is the magnetic field at the origin?

[26] J. A. Miranda, *Am. J. Phys.* **68**, 254 (2000).

(d) For a conic section with focus at the origin,

$$r(\theta) = \frac{p}{1 + e\cos\theta},$$

where p is the semilatus rectum (the y intercept) and e is the eccentricity ($e = 0$ for a circle, $0 < e < 1$ for an ellipse, $e = 1$ for a parabola). Show that the field is

$$B = \frac{\mu_0 I}{2p}$$

regardless of the eccentricity.[27]

Problem 5.52

(a) One way to fill in the "missing link" in Fig. 5.48 is to exploit the analogy between the defining equations for **A** (viz. $\nabla \cdot \mathbf{A} = 0$, $\nabla \times \mathbf{A} = \mathbf{B}$) and Maxwell's equations for **B** (viz. $\nabla \cdot \mathbf{B} = 0$, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$). Evidently **A** depends on **B** in exactly the same way that **B** depends on $\mu_0 \mathbf{J}$ (to wit: the Biot-Savart law). Use this observation to write down the formula for **A** in terms of **B**.

(b) The electrical analog to your result in (a) is

$$V(\mathbf{r}) = -\frac{1}{4\pi}\int \frac{\mathbf{E}(\mathbf{r}') \cdot \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau'.$$

Derive it, by exploiting the appropriate analogy.

! **Problem 5.53** Another way to fill in the "missing link" in Fig. 5.48 is to look for a magnetostatic analog to Eq. 2.21. The obvious candidate would be

$$\mathbf{A}(\mathbf{r}) = \int_{\mathcal{O}}^{\mathbf{r}} (\mathbf{B} \times d\mathbf{l}).$$

(a) Test this formula for the simplest possible case—uniform **B** (use the origin as your reference point). Is the result consistent with Prob. 5.25? You could cure this problem by throwing in a factor of $\frac{1}{2}$, but the flaw in this equation runs deeper.

(b) Show that $\int (\mathbf{B} \times d\mathbf{l})$ is *not* independent of path, by calculating $\oint (\mathbf{B} \times d\mathbf{l})$ around the rectangular loop shown in Fig. 5.63.

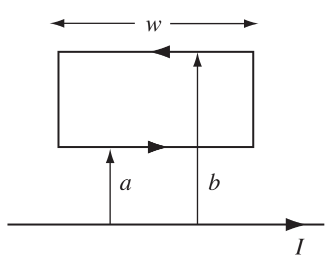


FIGURE 5.63

[27] C. Christodoulides, *Am. J. Phys.* **77**, 1195 (2009).

As far as I know,[28] the best one can do along these lines is the pair of equations

(i) $V(\mathbf{r}) = -\mathbf{r} \cdot \int_0^1 \mathbf{E}(\lambda \mathbf{r})\, d\lambda$,

(ii) $\mathbf{A}(\mathbf{r}) = -\mathbf{r} \times \int_0^1 \lambda \mathbf{B}(\lambda \mathbf{r})\, d\lambda$.

[Equation (i) amounts to selecting a *radial* path for the integral in Eq. 2.21; equation (ii) constitutes a more "symmetrical" solution to Prob. 5.31.]

(c) Use (ii) to find the vector potential for *uniform* **B**.

(d) Use (ii) to find the vector potential of an infinite straight wire carrying a steady current I. Does (ii) automatically satisfy $\nabla \cdot \mathbf{A} = 0$? [*Answer:* $(\mu_0 I/2\pi s)$ $(z\,\hat{\mathbf{s}} - s\,\hat{\mathbf{z}})$]

Problem 5.54

(a) Construct the scalar potential $U(\mathbf{r})$ for a "pure" magnetic dipole **m**.

(b) Construct a scalar potential for the spinning spherical shell (Ex. 5.11). [*Hint:* for $r > R$ this is a pure dipole field, as you can see by comparing Eqs. 5.69 and 5.87.]

(c) Try doing the same for the interior of a *solid* spinning sphere. [*Hint:* If you solved Prob. 5.30, you already know the *field*; set it equal to $-\nabla U$, and solve for U. What's the trouble?]

Problem 5.55 Just as $\nabla \cdot \mathbf{B} = 0$ allows us to express **B** as the curl of a vector potential ($\mathbf{B} = \nabla \times \mathbf{A}$), so $\nabla \cdot \mathbf{A} = 0$ permits us to write **A** itself as the curl of a "higher" potential: $\mathbf{A} = \nabla \times \mathbf{W}$. (And this hierarchy can be extended ad infinitum.)

(a) Find the general formula for **W** (as an integral over **B**), which holds when $\mathbf{B} \to \mathbf{0}$ at ∞.

(b) Determine **W** for the case of a *uniform* magnetic field **B**. [*Hint:* see Prob. 5.25.]

(c) Find **W** inside and outside an infinite solenoid. [*Hint:* see Ex. 5.12.]

Problem 5.56 Prove the following uniqueness theorem: If the current density **J** is specified throughout a volume $\mathcal{V}$, and *either* the potential **A** *or* the magnetic field **B** is specified on the surface $\mathcal{S}$ bounding $\mathcal{V}$, then the magnetic field itself is uniquely determined throughout $\mathcal{V}$. [*Hint:* First use the divergence theorem to show that

$$\int \{(\nabla \times \mathbf{U}) \cdot (\nabla \times \mathbf{V}) - \mathbf{U} \cdot [\nabla \times (\nabla \times \mathbf{V})]\}\, d\tau = \oint [\mathbf{U} \times (\nabla \times \mathbf{V})] \cdot d\mathbf{a},$$

for arbitrary vector functions **U** and **V**.]

Problem 5.57 A magnetic dipole $\mathbf{m} = -m_0\,\hat{\mathbf{z}}$ is situated at the origin, in an otherwise uniform magnetic field $\mathbf{B} = B_0\,\hat{\mathbf{z}}$. Show that there exists a spherical surface, centered at the origin, through which no magnetic field lines pass. Find the radius of this sphere, and sketch the field lines, inside and out.

[28] R. L. Bishop and S. I. Goldberg, *Tensor Analysis on Manifolds*, Section 4.5 (New York: Macmillan, 1968).

Problem 5.58 A thin uniform donut, carrying charge Q and mass M, rotates about its axis as shown in Fig. 5.64.

(a) Find the ratio of its magnetic dipole moment to its angular momentum. This is called the **gyromagnetic ratio** (or **magnetomechanical ratio**).

(b) What is the gyromagnetic ratio for a uniform spinning sphere? [This requires no new calculation; simply decompose the sphere into infinitesimal rings, and apply the result of part (a).]

(c) According to quantum mechanics, the angular momentum of a spinning electron is $\frac{1}{2}\hbar$, where $\hbar$ is Planck's constant. What, then, is the electron's magnetic dipole moment, in $\text{A} \cdot m^2$? [This semiclassical value is actually off by a factor of almost exactly 2. Dirac's relativistic electron theory got the 2 right, and Feynman, Schwinger, and Tomonaga later calculated tiny further corrections. The determination of the electron's magnetic dipole moment remains the finest achievement of quantum electrodynamics, and exhibits perhaps the most stunningly precise agreement between theory and experiment in all of physics. Incidentally, the quantity $(e\hbar/2m)$, where e is the charge of the electron and m is its mass, is called the **Bohr magneton**.]

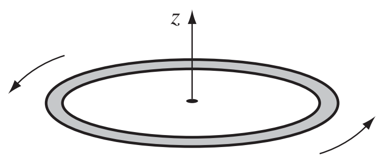


FIGURE 5.64

• **Problem 5.59**

(a) Prove that the average magnetic field, over a sphere of radius R, due to steady currents inside the sphere, is

$$\mathbf{B}_{\text{ave}} = \frac{\mu_0}{4\pi}\frac{2\mathbf{m}}{R^3}, \tag{5.93}$$

where $\mathbf{m}$ is the total dipole moment of the sphere. Contrast the electrostatic result, Eq. 3.105. [This is tough, so I'll give you a start:

$$\mathbf{B}_{\text{ave}} = \frac{1}{\frac{4}{3}\pi R^3}\int \mathbf{B}\, d\tau.$$

Write $\mathbf{B}$ as $(\boldsymbol{\nabla} \times \mathbf{A})$, and apply Prob. 1.61(b). Now put in Eq. 5.65, and do the surface integral first, showing that

$$\int \frac{1}{\mathscr{r}}\, d\mathbf{a} = \frac{4}{3}\pi \mathbf{r}'$$

(see Fig. 5.65). Use Eq. 5.90, if you like.]

(b) Show that the average magnetic field due to steady currents *outside* the sphere is the same as the field they produce at the center.

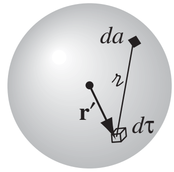


FIGURE 5.65

Problem 5.60 A uniformly charged solid sphere of radius R carries a total charge Q, and is set spinning with angular velocity ω about the z axis.

(a) What is the magnetic dipole moment of the sphere?

(b) Find the average magnetic field within the sphere (see Prob. 5.59).

(c) Find the approximate vector potential at a point (r, θ) where $r \gg R$.

(d) Find the *exact* potential at a point (r, θ) outside the sphere, and check that it is consistent with (c). [*Hint:* refer to Ex. 5.11.]

(e) Find the magnetic field at a point (r, θ) *inside* the sphere (Prob. 5.30), and check that it is consistent with (b).

Problem 5.61 Using Eq. 5.88, calculate the average magnetic field of a dipole over a sphere of radius R centered at the origin. Do the angular integrals first. Compare your answer with the general theorem in Prob. 5.59. Explain the discrepancy, and indicate how Eq. 5.89 can be corrected to resolve the ambiguity at $r = 0$. (If you get stuck, refer to Prob. 3.48.)

Evidently the *true* field of a magnetic dipole is[29]

$$\mathbf{B}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\right] + \frac{2\mu_0}{3}\mathbf{m}\delta^3(\mathbf{r}). \tag{5.94}$$

Compare the electrostatic analog, Eq. 3.106.

Problem 5.62 A thin glass rod of radius R and length L carries a uniform surface charge σ. It is set spinning about its axis, at an angular velocity ω. Find the magnetic field at a distance $s \gg R$ from the axis, in the xy plane (Fig. 5.66). [*Hint:* treat it as a stack of magnetic dipoles.] [*Answer:* $\mu_0\omega\sigma LR^3/4[s^2 + (L/2)^2]^{3/2}$]

[29] The delta-function term is responsible for the **hyperfine splitting** in atomic spectra—see, for example, D. J. Griffiths, *Am. J. Phys.* **50**, 698 (1982).

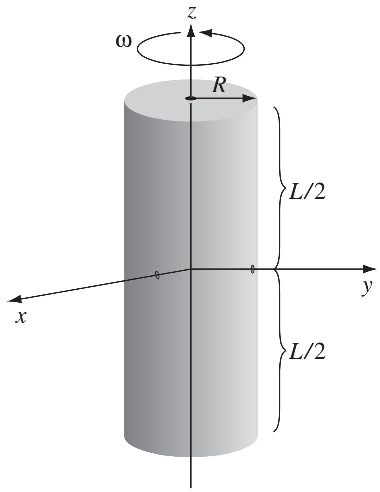


FIGURE 5.66